Both roots of Eq. (70) have negative real parts if the condition (24) is satisfied, and at least one root of Eq. (70) has a positive real part when the inequality (26) is satisfied. For the case of the Earth-pointing satellite, the effect of the postulated friction is thus simply to convert marginal stability into asymptotic stability. As for the case of the rotating satellite, the relationship between the stability of the solution $q = 0$ of Eq. (71) and the parameters g, h, and δ has been shown by V. G. Kotowski[5] to be such that, for $\delta > 0$, Fig. 6 is to be replaced with Fig. 8, in which the unshaded region once again corresponds to instability, and the shaded region is now associated with asymptotic stability. Thus it may be surmised that friction effects will be at worst, innocuous, and at best, helpful.

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AUGUST 1971 AIAA JOURNAL VOL. 9, NO. 8

Suboptimal Control of Linear Systems Derived from Models of Lower Dimension

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The problem considered is the following: given a linear dynamic system of order n, find the best model of order m, $m < n$, with which to derive a suboptimal control for the given system. The optimization problem covered is the infinite time, linear, output regulator problem with quadratic cost. The system and model outputs are characterized as elements of an appropriate Hilbert space, and the model output is constrained to be a projection of the system output. In this manner, the optimal model initial condition is expressed as a linear function of the system initial condition, and an algebraic expression is found for the modeling error which is minimized numerically. An example is presented wherein the pitch plane dynamics of a flexible-bodied rocket vehicle are modeled.

I. Introduction

SIMPLIFYING mathematical models of dynamic systems has traditionally relied heavily on the experience and ingenuity of the analyst. More recently, there has been considerable activity in the area of developing general techniques for simplifying, or approximating linear models of dynamic systems. These efforts have principally addressed the problem of modeling a system of linear homogeneous differential equations with a linear homogeneous model of lower order; usually constant coefficient. Modeling has been achieved using both frequency and time domain techniques, and the approach has varied in sophistication from pole removal to projection in a function space.[1,2]

This paper is concerned with the following problem: given a linear dynamic system of order n, find the best model of order m, $m < n$, with which to derive a suboptimal control for the given system. The optimization problem covered is the infinite time, linear, output regulator problem with quadratic cost. A model is derived such that the optimal control policy for the model is the "best" suboptimal control policy for the actual system.

Simplified models are important in applications of optimal control theory. For example, the synthesis of a stability augmentation system for a helicopter using the formalism of Murphy and Narenda involves solving successively several optimization problems with concurrent simulation of the system response to a variety of initial conditions.[3] If the dimension of the state vector is high, the computation time required by this procedure becomes excessive. The designer is forced to arbitrarily reduce the order of the model to an order compatible with the computer time allotted to the design. In other problems control-system simplicity is essential and a dynamic "observer" to reconstruct the components of the state vector not measured would be too complex. In this event a control policy that requires only a linear algebraic operation on the system outputs would have obvious advantages.

The modeling technique presented here provides a means of deriving a simple model and a suboptimal control policy that operates directly on the measured state variables. The approach to the modeling problem taken here is to characterize the system and model outputs as elements of a Hilbert space. The equations used in deriving the model are then obtained by constraining the model solutions to be projections of the

Received November 9, 1970; revision received April 9, 1971. This research was supported by the Joint Services Electronic Program (U.S. Army, U.S. Navy, and U.S. Air Force) under Grant AF-AFOSR-69-1622B.

Index Category: Navigation, Control, and Guidance Theory.

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solutions of the given system. The parameter matrices describing the model are found by using an iterative numerical technique to minimize an appropriate modeling cost functional.

II. Problem Statement

In order to prevent confusion over terminology, the terms "system" and "model" will be defined more precisely.

Definition 1

The "system" is a linearized mathematical description of a specified physical process. This description is as complete and accurate as the analyst can achieve, and is assumed to have the form

$$\dot{x} = Ax + Bu, \quad x(0) = x_o, \quad y = Cx \tag{1}$$

wherein x is an n-vector, u an r-vector and y an m-vector, with A, B, and C, nxn, nxr, and mxn, respectively. The initial condition x_o will be assumed to be a mean zero random variable with covariance matrix $\sigma^2 I$. It should be noted that the y vector is given a slightly different interpretation here than is usual. It will be assumed that the C matrix can be altered by the modeler. The individual components and also the dimension may be changed over some specific range. This is important in what follows since the dimension of the model is the same as the vector y, and the number of state variables measured can often be changed to achieve improved performance. In any given problem, the dimension of y, and hence, of the model, will be governed partly by physical considerations and partly by the ability of the designer to select a priori the most important components of the state vector.

It is assumed that the system is controllable and observable and that the cost functional

$$\hat{J} = \int_0^\infty [\langle y,Qy\rangle + \langle u,Ru\rangle]dt \tag{2}$$

is appropriate, where Q is an mxm positive definite matrix, and R is an rxr positive definite matrix.

Definition 2

The "model" of a given system is a simplified description of the underlying physical process. It is simpler in the sense that it is of lower order than the system defined previously, and therefore provides a less accurate picture of the physical process. The model is described by the set of equations

$$d\tilde{y}/dt = \tilde{A}\tilde{y} + \tilde{B}\tilde{u}, \quad \tilde{y}(0) = \tilde{y}_o \tag{3}$$

where $\tilde{y}$ is an m-vector, $\tilde{u}$ is an r-vector, and $\tilde{A}$ and $\tilde{B}$ are mxm and mxr, respectively. The model is assumed to be controllable and the cost functional associated with the model is

$$\tilde{J} = \int_0^\infty [\langle \tilde{y},Q\tilde{y}\rangle + \langle \tilde{u},R\tilde{u}\rangle]dt \tag{4}$$

where Q and R are the same weighting matrices used in the system cost. In this manner the components of the model state and control vectors are assigned the same importance as the corresponding components of the system output and control vectors.

The control policy for the system, Eq. (1), which minimizes the cost Eq. (2), is well known and is given by[4]

$$\hat{u}(t) = -R^{-1}B'Kx(t) \tag{5}$$

where K is the unique positive definite solution of the matrix equation

$$-A'K - KA + KBR^{-1}B'K - C'QC = 0 \tag{6}$$

The modeling problem is posed as follows. For the system described by Eq. (1), find a model described by Eq. (3) so that the response of the system using the optimal control policy of the model is as close to that of the model as possible. The degradation in performance attributable to the suboptimal controller used in the system will be measured by the functional

$$J_\Delta = \int_0^\infty [(y - \tilde{y})'Q(y - \tilde{y}) + (u - \tilde{u})'R(u - \tilde{u})]dt \tag{7}$$

where $\tilde{y}(t)$ is the optimal response of the model and $y(t)$ is obtained from Eq. (1) using the suboptimal controller. A model is called optimal if its dynamical matrices and initial condition are the minimizing elements of the functional

$$\min_{\tilde{A},\tilde{B}} E \left\{ \min_{\tilde{y}_0} E \left[\int_0^\infty (y - \tilde{y})'Q(y - \tilde{y}) + (u - \tilde{u})'R(u - \tilde{u})dt | x_o \right] \right\}$$

To motivate the problem as stated above, note that although the full solution is known, as a practical matter there are situations in which the optimal control may not be desirable. The actuating signals for the model and system are respectively

$$\tilde{u}(t) = -R^{-1}\tilde{B}'\tilde{K}\tilde{y}(t) \tag{8}$$

and

$$u(t) = -R^{-1}\tilde{B}'\tilde{K}y(t) \tag{9}$$

In most physical processes, the suboptimal feedback matrix $\tilde{K}$ operating on the output vector y will be substantially easier to implement than the optimal feedback matrix K operating on the state x. Not only is the dimension less in the suboptimal case, but this policy avoids those components of the state vector which are not readily accessible for use with feedback.

Another situation where the model will be of utility is in simulating the physical process on a computer. With the model as described previously, it is possible to explore the effects of various initial conditions on the model with the assurance that the actual system will have similar response. In the case of very large systems there may be modes which are very lightly coupled to the output, and which, consequently, have little effect on the cost. In this event a simple model which closely approximates the output of the system can give valuable information about system behavior with less expenditure of time and effort than a complete system simulation.

III. Solution of the Modeling Problem

It is well known that the solutions of a linear differential system form a linear space.[5] If the cost functional satisfies the inner product axioms, then it induces a norm on the set of solutions with the result that the set of all solutions to Eq. (1) constitutes a subspace of a Hilbert space. One of the most useful notions associated with Hilbert space theory is that of orthogonality. The purpose of this section will be to depict the model outputs as the orthogonal projections of the system outputs and to explore the implications thereof. Such a description is similar in spirit to the notion of the characteristic planes of the model and system.[6]

Recall that the optimal control with respect to $\tilde{J}$ given by Eq. (4) for the model, Eq. (3), has the form, Eq. (8), where $\tilde{K}$ is the positive definite solution of

$$-\tilde{A}'\tilde{K} - \tilde{K}\tilde{A} + \tilde{K}\tilde{B}R^{-1}\tilde{B}'\tilde{K} - Q = 0 \tag{10}$$

When this is substituted for $\tilde{u}$ in Eq. (3) the resultant equation is

$$d\tilde{y}/dt = (\tilde{A} - \tilde{B}R^{-1}\tilde{B}'\tilde{K})\tilde{y}, \quad \tilde{y}(o) = \tilde{y}_o \tag{11}$$

The optimal model response is given by

$$\tilde{y}(t) = e^{(\tilde{A} - \tilde{B}R^{-1}\tilde{B}'\tilde{K})t}\tilde{y}_o \tag{12}$$

and the associated cost is

$$\tilde{J} = \int_0^\infty \tilde{y}'\Omega\tilde{y}dt \tag{13}$$

where

$$\Omega = [Q + \tilde{K}\tilde{B}R^{-1}\tilde{B}'\tilde{K}] \tag{14}$$

The matrix Ω is positive definite by the assumptions on Q and R, so that the aforementioned integral satisfies the inner product axioms, and induces a norm on the set of square integral m-dimensional functions on $[0, \infty]$[5]

$$\|y\|_\Omega^2 = \langle y,y \rangle = \int_0^\infty y'\Omega y dt \tag{15}$$

The assumptions on the model guarantee that the model with control given by Eq. (8) is stable. Hence, for any initial condition $\tilde{y}_o$; $\|\tilde{y}\|_\Omega < \infty$. Further, the columns of the model transition matrix, Eq. (12), are linearly independent, so that every model solution lies in an m-dimensional subspace of the Hilbert space.

Using the optimal control policy for the model in the system, the closed-loop equation for the system becomes

$$\dot{x} = Ax - BR^{-1}\tilde{B}'\tilde{K}y = (A - BR^{-1}\tilde{B}'\tilde{K}C)x, \quad x(o) = x_o \tag{16}$$

Thus, the output of the suboptimally controlled system has the form

$$y(t) = Ce^{(A - BR^{-1}\tilde{B}'\tilde{K}C)t}x_o \tag{17}$$

The system cost functional, Eq. (2), becomes

$$\hat{J} = \int_0^\infty [y'Qy + y'\tilde{K}\tilde{B}R^{-1}\tilde{B}'\tilde{K}y]dt = \|y\|_\Omega^2 \tag{18}$$

The metric induced by Ω provides a means of measuring the "distance" between system and model solutions, and thus is a natural means of evaluating how well a given model output simulates the corresponding system output. From Eqs. (7–9)

$$J_\Delta = \int_0^\infty (y - \tilde{y})(Q + \tilde{K}\tilde{B}R^{-1}\tilde{B}'\tilde{K})(y - \tilde{y})dt \tag{19}$$

$$= \langle (y - \tilde{y}), (y - \tilde{y}) \rangle_\Omega = \|y - \tilde{y}\|_\Omega^2 \tag{20}$$

Note that while both y and $\tilde{y}$ have m components, the system matrix $Ce^{(A - BR^{-1}\tilde{B}'\tilde{K}C)t}$ has n linearly independent columns, under the assumption that the system is observable. Hence, it is possible to treat the model output space as an m-dimensional subspace of the system output space. The error, or distance between model and system output will then assume its minimum when it is orthogonal to the model output, i.e., when

$$\langle (y - \tilde{y}), \tilde{y} \rangle_\Omega = 0 \tag{21}$$

for all y_o.

Equation (21) may also be written as

$$\langle y,\tilde{y} \rangle_\Omega = \langle \tilde{y},\tilde{y} \rangle_\Omega \tag{22}$$

Since the Hilbert Space, as defined, is real, then

$$\langle y,\tilde{y} \rangle_\Omega = \langle \tilde{y},y \rangle_\Omega \tag{23}$$

Another result of importance is that when the error is orthogonal to $\tilde{y}$, then

$$\|y - \tilde{y}\|_\Omega^2 = \|y\|_\Omega^2 - \|\tilde{y}\|_\Omega^2 \tag{24}$$

To explore the implications of the orthogonality principle, we write

$$y(t) = Ce^{(A - BR^{-1}\tilde{B}'\tilde{K}C)t}x_o = C\Phi(t)x_o \tag{25}$$

and

$$\tilde{y}(t) = e^{(\tilde{A} - \tilde{B}R^{-1}\tilde{B}'\tilde{K})t}\tilde{y}_o = \tilde{\Phi}(t)\tilde{y}_o \tag{26}$$

Note that any $\tilde{y}(t)$ can be written as a linear combination of the columns of $\tilde{\Phi}(t)$. Thus, if

$$\tilde{\Phi}(t) = (g_1, g_2, \ldots, g_m) \tag{27}$$

then

$$\tilde{y}(t) = \sum_{i=1}^m \alpha_i g_i \tag{28}$$

where

$$y_o = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_m \end{bmatrix} \tag{29}$$

A stronger statement than Eq. (21) may be made. Not only is the error $y - \tilde{y}$ orthogonal to $\tilde{y}$, it is also orthogonal to each of the g_i; i.e.[5]

$$\langle (y - \tilde{y}), g_i \rangle_\Omega = 0, \quad i = 1,2, \ldots m \tag{30}$$

Hence

$$\langle y,g_i \rangle_\Omega = \langle \tilde{y},g_i \rangle_\Omega \quad i = 1,2, \ldots m \tag{31}$$

or

$$\langle g_i,y \rangle_\Omega = \langle g_i,\tilde{y} \rangle_\Omega \quad i = 1,2, \ldots m \tag{32}$$

since the functions are real.

This relationship can be used to determine the optimal condition $\tilde{y}_o$ for the model corresponding to any given initial condition x_o on the system. Thus

$$\langle g_i,\tilde{y} \rangle_\Omega = \int_0^\infty g_i'\Omega\tilde{y}dt = \int_0^\infty g_i'\Omega\tilde{\Phi}(t)dt\tilde{y}_o \quad i = 1, 2, \ldots m \tag{33}$$

By noting the definition of the g_i, and taking account of their ordering, it is possible to write the inner products for all $i = 1, 2, \ldots, m$ compactly in the matrix equation

$$\begin{bmatrix} \langle g_1,\tilde{y} \rangle_\Omega \\ \langle g_2,\tilde{y} \rangle_\Omega \\ \cdot \\ \cdot \\ \cdot \\ \langle g_m,\tilde{y} \rangle_\Omega \end{bmatrix} = \int_0^\infty \tilde{\Phi}'(t)\Omega\tilde{\Phi}(t)dt\tilde{y}_o = G\tilde{y}_o \tag{34}$$

where

$$G = \int_0^\infty \tilde{\Phi}(t)\Omega\tilde{\Phi}(t)dt \tag{35}$$

Similarly, evaluating the left hand side of Eq. (32) gives

$$\begin{bmatrix} \langle g_1,y \rangle_\Omega \\ \langle g_2,y \rangle_\Omega \\ \cdot \\ \cdot \\ \langle g_m,\tilde{y} \rangle_\Omega \end{bmatrix} = \int_0^\infty \tilde{\Phi}'(t)\Omega C\Phi(t)dtx_o = Fx_o \tag{36}$$

where

$$F = \int_0^\infty \tilde{\Phi}'(t)\Omega C\Phi(t)dt \tag{37}$$

Thus [see also Eq. (2) of Ref. 2]

$$G\tilde{y}_o = Fx_o \tag{38}$$

This relationship may be carried one step further. Into the expression for the optimal model cost

$$\tilde{J} = \|\tilde{y}\|_\Omega^2 = \int_0^\infty \tilde{y}'\Omega\tilde{y}dt \tag{39}$$

we substitute for $\tilde{y}$ the relation (26). Then

$$\tilde{J} = \tilde{y}_o' \int_0^\infty \tilde{\Phi}'(t)\Omega\tilde{\Phi}(t)dt\tilde{y}_o \tag{40}$$

But, from Eq. (35), this may be written

$$\tilde{J} = \tilde{y}_o'G\tilde{y}_o \tag{41}$$

It is known that the optimum cost is[4]

$$\tilde{J} = \tilde{y}_o'\tilde{K}\tilde{y}_o \tag{42}$$

Hence

$$G = \tilde{K} \tag{43}$$

and Eq. (38) becomes

$$\tilde{K}\tilde{y}_o = Fx_o \tag{44}$$

But $\tilde{K}$ is the positive definite solution of Eq. (9), and is therefore invertible. The optimal model initial condition corresponding to a given system initial condition is

$$\tilde{y}_o = \tilde{K}^{-1}Fx_o \tag{45}$$

It is now possible to evaluate the modeling error, Eq. (19), in terms of system and model parameters. From Eq. (23) we have

$$\|y - \tilde{y}\|^2 = \|y\|_\Omega^2 - \|\tilde{y}\|_\Omega^2 \tag{46}$$

From Eqs. (39) and (43)

$$\|\tilde{y}\|_\Omega^2 = \tilde{y}_o'\tilde{K}\tilde{y}_o = x_o'F\tilde{K}^{-1}Fx_o \tag{47}$$

Evaluating $\|y\|_\Omega^2$ gives

$$\|y\|_\Omega^2 = \langle y,y\rangle_\Omega = \int_0^\infty y'\Omega y dt = x_o'Mx_o \tag{48}$$

where

$$M = \int_0^\infty \Phi'(t)C'\Omega C\Phi(t)dt \tag{49}$$

Thus, for a specified initial condition x_o the modeling error may be written

$$\|y - \tilde{y}\|_\Omega^2 = x_o'(M - F'\tilde{K}^{-1}F)x_o \tag{50}$$

The expected value of this modeling error is

$$E\{J_\Delta\} = \sigma^2 \operatorname{Tr}(M - F'\tilde{K}^{-1}F) \tag{51}$$

where $\operatorname{Tr}(\cdot)$ denotes the trace of a matrix.

IV. Reduction to Algebraic Form

At this point the modeling problem has, in principle, been solved; the mean modeling error has been expressed as a function of the model parameter matrices, $\tilde{A}$ and $\tilde{B}$, and it can be minimized numerically using an appropriate algorithm. Before doing so, however, it is advantageous to express the definitions of M and F in a form more suitable for computer solution. It is shown in Ref. 7 that under fairly general conditions if the relation

$$X = -\int_0^\infty e^{At}Ce^{Bt}dt \tag{52}$$

holds, then X is the unique solution of the algebraic equation

$$AX + XB = C \tag{53}$$

where A is an nxn matrix, B is an mxm matrix, and X and C are nxm matrices. It is thus possible to express the equations defining F and M in equivalent algebraic form. From Eqs. (37) and (49) we have

$$(\tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{B}'\tilde{K})'F + F(A - BR^{-1}\tilde{B}'\tilde{K}C) = -\Omega C \tag{54}$$

and

$$(A - BR^{-1}\tilde{B}'\tilde{K}C)'M + M(A - BR^{-1}\tilde{B}'\tilde{K}C) = -C'\Omega C \tag{55}$$

These equations along with the matrix equation for $\tilde{K}$ are the basic elements in the algorithm for evaluating the modeling cost $E\{J_\Delta\}$. Since the σ^2 is fixed, minimizing the trace of $(M - F'\tilde{K}^{-1}F)$ is equivalent to minimizing the expected value of the modeling error. Using the properties of the trace, the expression for the normalized cost may now be written in the computationally convenient form

$$J^* = \operatorname{Tr}(M) - \operatorname{Tr}(\tilde{K}^{-1}FF') \tag{56}$$

Using Eqs. (10), (54), and (55), it is possible to find the $\tilde{A}$ and $\tilde{B}$ which minimize J^*. Due to the complex functional dependence of J^* on $\tilde{A}$ and $\tilde{B}$ it is not possible, in general, to solve these equations explicitly. Instead, a gradient search technique on a digital computer has been used. In order to facilitate the numerical minimization of J^*, a set of variational equations are derived in the next section.

V. Variational Equations

Although not as commonly used as the conventional gradient, the derivative of a scalar function of a matrix with respect to the elements of the matrix has been treated by several authors.[8,9] Basically, if a scalar f is a function of an rxn matrix X, with elements x_{ij}, then its gradient matrix is an rxn matrix whose ijth element is given by $\partial f/\partial x_{ij}$. In the numerical minimization of J^*, it is convenient to use the finite difference approximation to the gradient matrix having as its ijth element

$$\partial f(X)/\partial x_{ij} \cong [f(X + \Delta X_{ij}) - f(X)]/\delta x_{ij} \tag{57}$$

where ΔX_{ij} is an rxn matrix having as its elements all zeros except for its ijth element, which is δx_{ij}. Finding the approximate gradient matrix of J^* is a two-stage problem, since J^* is explicitly a function of the intermediate matrices M, F, and $\tilde{K}$, which, in turn, depend on $\tilde{A}$ and $\tilde{B}$.

To derive the variational equations, either $\tilde{A}$ is perturbed by the matrix δA which is mxm with all zero elements except δa_{ij}, or $\tilde{B}$ is perturbed by an mxr matrix δB which has all zero elements except δb_{kl}. Substituting these into the matrix equation for $\tilde{K}$ yields

$$-(\tilde{A} + \delta A)'(\tilde{K} + \delta K) - (\tilde{K} + \delta\tilde{K})(\tilde{A} + \delta A) + (\tilde{K} + \delta K)(\tilde{B} + \delta B)R^{-1}(\tilde{B} + \delta B)'(\tilde{K} + \delta K) - Q = 0 \tag{58}$$

Carrying out the indicated multiplications and retaining first order perturbations yields

$$-\tilde{A}'\tilde{K} - \delta A'\tilde{K} - \tilde{A}'\delta K - \tilde{K}\tilde{A} - \delta K\tilde{A} - \tilde{K}\delta A + \tilde{K}\tilde{B}R^{-1}\tilde{B}^{-1}\tilde{K} + \tilde{K}\tilde{B}R^{-1}\tilde{B}'\delta\tilde{K} + \tilde{K}\tilde{B}R^{-1}\delta B'\tilde{K} + \tilde{K}\delta\tilde{B}R^{-1}\tilde{B}'\tilde{K} + \delta K\tilde{B}R^{-1}\tilde{B}'\tilde{K} - Q + o(\delta) = 0 \tag{59}$$

where

$$\lim_{\delta\to 0} \frac{o(\delta)}{\delta} = 0 \tag{60}$$

Carrying out the algebraic simplification of Eq. (59) gives, to a first order in δK

$$(\tilde{A} - \tilde{B}R^{-1}\tilde{B}'\tilde{K})'\delta K + \delta K(\tilde{A} - \tilde{B}R^{-1}\tilde{B}'\tilde{K}) = -\{(\delta A - \delta BR^{-1}\tilde{B}'\tilde{K})'\tilde{K} + \tilde{K}(\delta A - \delta BR^{-1}\tilde{B}'\tilde{K})\} \tag{61}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} -0.21053 & -0.10526 & -0.0007378 & 0.0 & 0.0706 & 0.0 \\ 1.0 & -0.03537 & -0.0001180 & 0.0 & 0.0004 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -605.16 & -4.92 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -3906.25 & -12.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} -7.211 \\ -0.05232 \\ 0.0 \\ 794.7 \\ 0.0 \\ -448.5 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.000334 & 0.0 & -0.007728 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

Fig. 1 System of equations describing flexible dynamics of rocket vehicle.

Note that this variational equation for δK has the same form as the equation defining F and M. In a similar manner, equations have been derived defining δF and δM. They are, to a first order

$$[\tilde{A} - \tilde{B}R^{-1}\tilde{B}'\tilde{K}]'\delta F + \delta F[A - BR^{-1}\tilde{B}'\tilde{K}] = -\{[\tilde{K}\tilde{B}R^{-1}(\delta B'\tilde{K} + \tilde{B}'\delta K) + (\tilde{K}\delta B + \delta K\tilde{B})R^{-1}\tilde{B}'\tilde{K}]C + [\delta A - \tilde{B}R^{-1}(\tilde{B}'\delta K + \delta B'\tilde{K}) - \delta BR^{-1}\tilde{B}'\tilde{K}]'F - F[\tilde{B}R^{-1}(\tilde{B}'\delta K + \delta B'\tilde{K})]C\} \quad (62)$$

and

$$[A - BR^{-1}\tilde{B}'\tilde{K}C]'\delta M + \delta M[A - BR^{-1}\tilde{B}'\tilde{K}C] = \{[BR^{-1}(\delta B'\tilde{K} + \tilde{B}\delta K)C]'M + M[BR^{-1}(\delta B'\tilde{K} + \tilde{B}'\delta K)C] - C'[\tilde{K}\tilde{B}R^{-1}(\delta B'\tilde{K} + \tilde{B}\delta K) + (\tilde{K}\delta B + \delta K\tilde{B})R^{-1}\tilde{B}'\tilde{K}]C\} \quad (63)$$

By use of the same variational technique the increment in the cost is found as a function of δK, δF, and δM. Thus

$$(J^* + \delta J) = \text{Tr}\{(M + \delta M) - (F + \delta F)(K + \delta K)^{-1}(F + \delta F)\} \quad (64)$$

But

$$(\tilde{K} + \delta K)^{-1} = (\tilde{K}^{-1} - \tilde{K}^{-1}\delta K\tilde{K}^{-1}) + o(\delta K) \quad (65)$$

Simplifying, and utilizing the properties of the trace yields finally

$$\delta J = \text{Tr}(\delta M) - \text{Tr}\{\tilde{K}^{-1}(2\delta F - \delta K\tilde{K}^{-1}F)F'\} \quad (66)$$

By sequentially perturbing the elements of $\tilde{A}$ and then $\tilde{B}$ in the preceding variational equations, it is possible to derive the gradient matrices for the system. The elements of these gradient matrices may be programed in vector form for use with one of the methods available in the literature to achieve rapid convergence to the $\tilde{A}$ and $\tilde{B}$ which minimize the modeling cost functional, J^*.[10]

VI. Example

In order to demonstrate the applicability of the preceding theoretical development to an engineering problem, an example was chosen in which the system of equations to be modeled represent the pitch plane dynamics of a flexible bodied rocket vehicle. The variables describing the rigid body motion of the vehicle are angle of attack, α, and pitch rate, $\dot{\theta}$. These are controlled by deflection of a rocket nozzle or movable flaps. The control deflection is denoted by δ. The small amplitude oscillations of the vehicle due to its flexibility are normally called the bending modes. Each of these is characterized by a second-order differential equation with a small damping coefficient. It is usually necessary to consider the behavior of the first two or three bending modes when designing a control system for such a vehicle. When the first two bending modes are considered along with the rigid body dynamics, a sixth order system results. This example is considered to be a relatively severe test of the modeling procedure due to the low damping of the bending modes.

The system to be modeled is shown in Fig. 1. The physical significance of the variables is as follows

State Vector			
x_1:	$\dot{\theta}$	pitch rate	rad/sec
x_2:	α	angle of attack	rad
x_3:	ξ_1	1st mode deflection	rad
x_4:	$\dot{\xi}_1$	1st mode deflection rate	rad/sec
x_5:	ξ_2	2nd mode deflection	rad
x_6:	$\dot{\xi}_2$	2nd mode deflection rate	rad/sec
Control			
u:	δ	control deflection	rad
Observables			
y_1:		pitch rate mixed with bending rates	rad/sec
y_2:		angle of attack	rad

The eigenvalues of the system matrix shown in Fig. 1 are: $\lambda_1,\lambda_2 = -0.1229 \pm j\,0.3124$; $\lambda_3,\lambda_4 = -2.46 \pm j\,24.6$; $\lambda_5,\lambda_6 = -6.25 \pm j\,62.5$. A digital computer modeling program embodying the modeling equations was used to derive an optimal model and associated set of feedback gains for the system of Fig. 1. The resultant model for $Q = R = I$ is given by

$$\frac{d}{dt}\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} -1.415 & -0.75758 \\ 0.75081 & -0.25730 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} + \begin{bmatrix} -6.7317 \\ -0.0002 \end{bmatrix} \tilde{u}$$

while the feedback gain matrix is

$$\tilde{K} = \begin{bmatrix} 0.12984 & 0.08748 \\ 0.08748 & 1.01153 \end{bmatrix}$$

The suboptimally controlled system using this feedback gain matrix and the projection matrix F are shown in Fig. 2. The optimally controlled model is given by

$$\frac{d}{dt}\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} -7.29874 & -4.72256 \\ 0.75074 & -0.25734 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}$$

The eigenvalues of the optimized model are $\lambda_1 = -6.75292$; $\lambda_2 = -0.80316$. While the eigenvalues of the suboptimally controlled system are: $\lambda_1 = -5.63949$; $\lambda_2 = -0.79236$; $\lambda_3,\lambda_4 = -1.3779 \pm j\,24.87508$; $\lambda_5,\lambda_6 = -4.73120 \pm j\,62.24081$. The comparative time response of system and

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} -6.5132 & -4.3525 & -0.0074 & -0.0211 & 0.0706 & 0.0487 \\ 0.9543 & -0.0662 & -0.0001 & -0.0002 & 0.0004 & 0.0004 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 694.5925 & 468.0778 & -605.16 & -2.6001 & 0.0 & -5.3678 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ -392.0029 & -264.1662 & 0.0 & -1.3093 & -3906.25 & -9.4706 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$F = \begin{bmatrix} 0.1432 & 0.0961 & -0.0026 & 0.0 & 0.0066 & 0.0 \\ 0.1283 & 1.0643 & 0.0013 & 0.0 & -0.0027 & 0.0 \end{bmatrix}$$

Fig. 2 Suboptimally controlled system, and projection matrix.

model are shown in Fig. 3 for various sets of initial conditions. It may be observed that the model outputs follow the low-frequency components of the system outputs rather well.

It should be noted that there are other approaches to the problem of reducing the order of the system model. One such method was brought to the attention of the authors by a reviewer of this paper. By introducing dynamic operations in series with the actuating signal it is possible to incorporate residual stiffness effects of the first two deflection modes into the rigid body representation.[11,12] Such an approach has

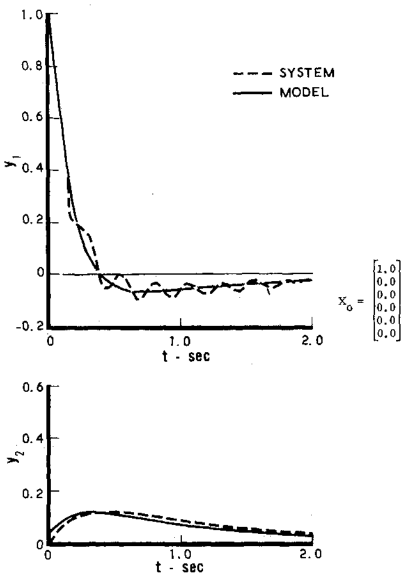

Fig. 3a

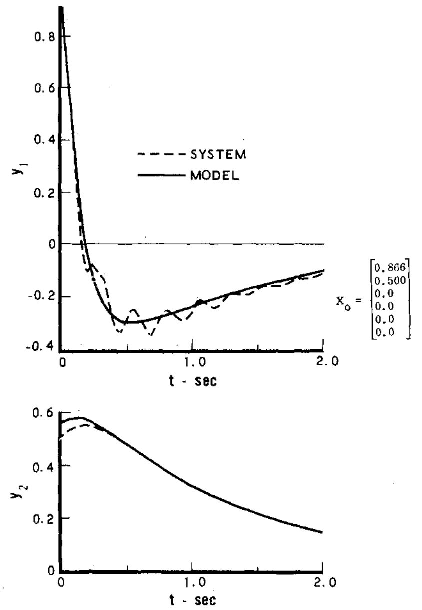

Fig. 3b

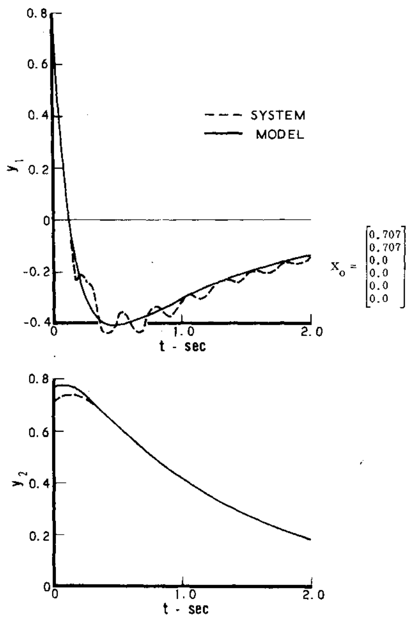

Fig. 3c

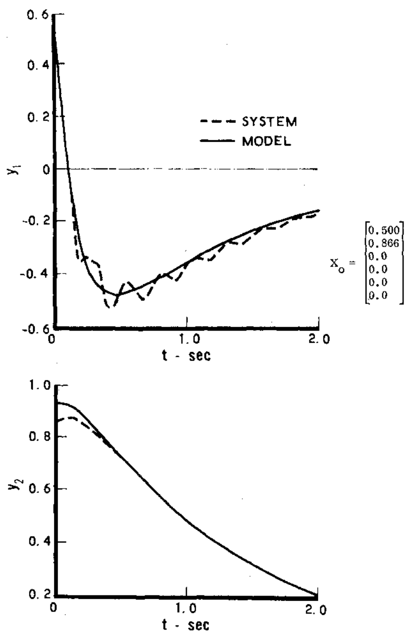

Fig. 3d

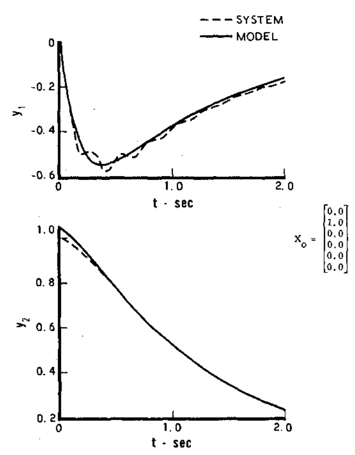

Fig. 3e

Fig. 3 Comparison of system and model response for the system initial condition X_o.

several advantages. It is much simpler computationally, and when employed by an experienced designer it gives a physical insight into the way the neglected modes of response effect closed loop performance.

It has the usual limitations introduced by the approximations inherent in the procedure. For systems without the natural modal decomposition displayed by this example, the influence of these approximations may be difficult to evaluate. The open loop poles of a model derived from the techniques of Refs. 11 and 12 differ significantly from those derived here with the result that an optimal control derived on the basis of the former model would not necessarily be similar to that presented here. It can be shown, however, that if the controller presented in this example is used with either model, the closed loop poles are nearly the same.

VII. Conclusion

A procedure for evaluating a simple model of a complicated, linear dynamical system has been presented. The model is selected to insure that the actual system response is as close as possible to that attained by the optimally controlled model. Particularly when vehicle trajectories must be obtained by simulation for a wide variety of initial conditions, the simple model will help reduce the computational requirements while providing insight into the important dynamic properties of the system.

The assumption on the initial condition vector x_o can be changed and similar results derived. If x_o is fixed or constrained to lie within a convex set, the preceding analysis is still applicable for the most part. The most important change is in J^* (see Eq. 56)) which would now be related to the eigenvalues of $(M - F\tilde{K}F')$. The same numerical methods can be used in this event, but the solution is more difficult.

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